MEAN-SQUARE ERROR CALCULATIONS FOR THE SUBBAND ADAPTIVE FILTER SYSTEM

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ABSTRACT

In this paper we develop theoretical equations for calculating the reconstructed mean-square error (MSE), $\mathrm{E}\left[\left|e^2\right|\right]$ for the subband adaptive filter system [1] in Fig. 1 under zero mean, unit variance white Gaussian input. These systems are seeing increased use in acoustic echo cancellation applications and by performing the calculations, we are potentially able to evaluate and compare performance for various design parameters and various modifications to the basic design.

1. INTRODUCTION

The acoustic echo cancellation problem has been studied for a number of years and currently, the oversampled, subband adaptive filter approach seems to be the widely recognized method to combat the echo [2]. In designing these systems, one is faced with many choices such as analysis/synthesis filter design [3], the number of subbands, oversampling factor, length of adaptive filters, etc.... The ability to characterize the performance of these systems for the various design parameters helps in making optimal choices and trade offs.

In this paper we calculate the reconstructed MSE for the subband adaptive filtering system in Fig. 1 under zero mean, unit variance white Gaussian input. In these calculations, we assume the LMS algorithm is used to update the adaptive filters but no assumptions are made on the number of subbands or whether they are critically sampled or oversampled. Furthermore, we do not make any assumptions on the analysis and synthesis filters other than that they are FIR.

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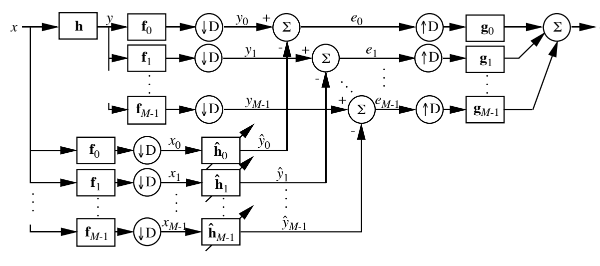

Figure 1: Subband adaptive filtering system with reference analysis filters and unknown system commuted.

2. CALCULATIONS

We begin by assuming that the analysis and synthesis filters ($\mathbf{f}_0,\cdots,\mathbf{f}_{M-1}$ and $\mathbf{g}_0,\cdots,\mathbf{g}_{M-1}$) are each of length L, the adaptive filters ($\hat{\mathbf{h}}_0,\cdots,\hat{\mathbf{h}}_{M-1}$) are each of length N, and the unknown system ($\mathbf{h}$) is of length H and is real. In the calculations, we treat signals and filters as column vectors with the appropriate dimensions. For convenience, we commute the reference analysis filters with the unknown system and establish notation for the various signals as in Fig. 2.

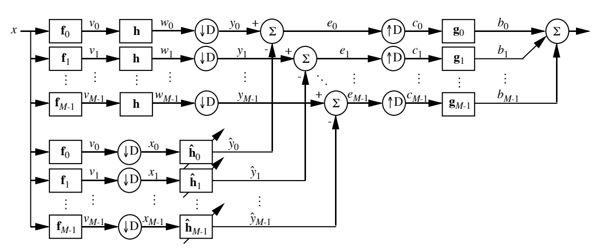

Figure 2: Subband adaptive filtering system with reference analysis filters and unknown system commuted.

We begin the calculation by defining the reconstructed MSE at sample index k as:

$$\xi_k \equiv \mathrm{E}\left[e_k e_k^*\right]. \tag{1}$$

Substituting

$$e_k = b_{0,k} + \cdots + b_{M-1,k} \tag{2}$$

into (1) we have:

$$\xi_k = \mathrm{E}\left[\left(b_{0,k} + \cdots + b_{M-1,k}\right)\left(b_{0,k} + \cdots + b_{M-1,k}\right)^*\right]. \tag{3}$$

Applying the linearity property of the expectation, we examine a single term, $\mathrm{E}\left[b_{m,k} b_{n,k}^*\right]$, from (3) which we rewrite as:

$$\mathrm{E}\left[b_{m,k} b_{n,k}^*\right] = \mathrm{E}\left[\mathbf{g}_m^T \mathbf{c}_{m,k} \mathbf{c}_{n,k}^H \mathbf{g}_n^*\right] \tag{4}$$

where

$$\mathbf{g}_m = \left[g_{m,0} \ \cdots \ g_{m,L-1}\right]^T, \tag{5}$$

$$\mathbf{c}_{m,k} = \left[c_{m,k} \ \cdots \ c_{m,k-L+1}\right]^T, \tag{6}$$

$$c_{m,k} = \begin{cases} e_{\frac{k}{D}}, & k \bmod D \equiv 0 \\ 0, & \text{otherwise} \end{cases}, \tag{7}$$

and ^{H} is the complex conjugate transpose. Substituting (7) into (6) and using the result in (4) and letting $q = k \bmod D$ we have:

$$\begin{aligned} \mathrm{E}\left[b_{m,k} b_{n,k}^*\right] &= \mathbf{g}_m^T \, \mathrm{E}\left[\begin{bmatrix} \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\} q \\ e_{m,\frac{k-q}{D}} \\ \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\} D-1 \\ e_{m,\frac{k-q-D}{D}} \\ \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\} D-1 \\ e_{m,\frac{k-q-2D}{D}} \\ \vdots \end{bmatrix} \left[\underbrace{0 \cdots 0}_{q} \ e_{n,\frac{k-q}{D}}^* \ \underbrace{0 \cdots 0}_{D-1} \ e_{n,\frac{k-q-D}{D}}^* \ \underbrace{0 \cdots 0}_{D-1} \ e_{n,\frac{k-q-2D}{D}}^* \ \cdots\right]\right] \mathbf{g}_n^* \\ &= \mathbf{g}_{m,q}^{T\,D} \, \mathrm{E}\left[\mathbf{e}_{m,\frac{k-q}{D}} \mathbf{e}_{n,\frac{k-q}{D}}^H\right] \mathbf{g}_{n,q}^{*\,D} \end{aligned} \tag{8}$$

where

$$\mathbf{g}_{m,q}^{D} = \left[g_{m,q} \ \ g_{m,q+D} \ \ g_{m,q+2D} \ \cdots \ g_{m,q+\left\lfloor \frac{L-1-q}{D} \right\rfloor D}\right]^T \tag{9}$$

and

$$\mathbf{e}_{m,\frac{k-q}{D}} = \left[e_{m,\frac{k-q}{D}} \ e_{m,\frac{k-q-D}{D}} \ e_{m,\frac{k-q-2D}{D}} \cdots e_{m,\frac{k-q-\left\lfloor \frac{L-1-q}{D} \right\rfloor D}{D}}\right]^T. \tag{10}$$

The subband error at sample index k is computed as:

$$e_{m,k} = y_{m,k} - \hat{y}_{m,k} \tag{11}$$

and substituting (11) into (10) and the result into (8) yields:

$$\mathrm{E}\left[b_{m,k} b_{n,k}^*\right] = \mathbf{g}_{m,q}^{T\,D} \, \mathrm{E}\left[\left(\mathbf{y}_{m,\frac{k-q}{D}} - \hat{\mathbf{y}}_{m,\frac{k-q}{D}}\right)\left(\mathbf{y}_{n,\frac{k-q}{D}} - \hat{\mathbf{y}}_{n,\frac{k-q}{D}}\right)^H\right] \mathbf{g}_{n,q}^{*\,D} \tag{12}$$

where

$$\mathbf{y}_{m,\frac{k-q}{D}} = \left[y_{m,\frac{k-q}{D}} \ y_{m,\frac{k-q-D}{D}} \ y_{m,\frac{k-q-2D}{D}} \cdots y_{m,\frac{k-q-\left\lfloor \frac{L-1-q}{D} \right\rfloor D}{D}}\right]^T \tag{13}$$

and

$$\hat{\mathbf{y}}_{m,\frac{k-q}{D}} = \left[\hat{y}_{m,\frac{k-q}{D}} \ \hat{y}_{m,\frac{k-q-D}{D}} \ \hat{y}_{m,\frac{k-q-2D}{D}} \cdots \hat{y}_{m,\frac{k-q-\left\lfloor \frac{L-1-q}{D} \right\rfloor D}{D}}\right]^T. \tag{14}$$

We next substitute:

$$\begin{aligned} y_{m,\frac{k-q}{D}} &= w_{m,k-q} \\ &= \mathbf{h}^T \underline{\underline{\mathbf{v}}}_{m,k-q} \end{aligned} \tag{15}$$

into (13) where

$$\mathbf{h} = \left[h_0 \ \cdots \ h_{H-1}\right]^T \tag{16}$$

and

$$\underline{\underline{\mathbf{v}}}_{m,k-q} = \left[v_{m,k-q} \ \cdots \ v_{m,k-q-H+1}\right]^T \tag{17}$$

and we also substitute

$$\begin{aligned} \hat{y}_{m,\frac{k-q}{D}} &= \hat{\mathbf{h}}_{m,\frac{k-q}{D}}^H \mathbf{x}_{m,\frac{k-q}{D}} \\ &= \hat{\mathbf{h}}_{m,\frac{k-q}{D}}^H \underline{\mathbf{v}}_{m,k-q}^{D} \end{aligned} \tag{18}$$

into (14) where $\hat{\mathbf{h}}_{m,k}$ is the expected value of the adaptive filter for subband m at sample index k:

$$\hat{\mathbf{h}}_{m,k} = \left[\hat{h}_{m,k,0} \ \cdots \ \hat{h}_{m,k,N-1}\right]^T, \tag{19}$$

$$\mathbf{x}_{m,\frac{k-q}{D}} = \left[x_{m,\frac{k-q}{D}} \ \cdots \ x_{m,\frac{k-q}{D}-N+1}\right]^T, \tag{20}$$

and

$$\underline{\mathbf{v}}_{m,k-q}^{D} = \left[v_{m,k-q} \ v_{m,k-q-D} \cdots v_{m,k-q-D(N-1)}\right]^T. \tag{21}$$

The calculation for $\hat{\mathbf{h}}_{m,k}$ is given in Section 3. Substituting (15) into (13) and (18) into (14) and the results into (12) we arrive at:

$$\begin{aligned} \mathrm{E}\left[b_{m,k} b_{n,k}^*\right] = \mathbf{g}_{m,q}^{T\,D} \, \mathrm{E}\Bigg[& \begin{bmatrix} \mathbf{h}^T \underline{\underline{\mathbf{v}}}_{m,k-q} - \hat{\mathbf{h}}_{m,\frac{k-q}{D}}^H \underline{\mathbf{v}}_{m,k-q}^{D} \\ \mathbf{h}^T \underline{\underline{\mathbf{v}}}_{m,k-q-D} - \hat{\mathbf{h}}_{m,\frac{k-q-D}{D}}^H \underline{\mathbf{v}}_{m,k-q-D}^{D} \\ \vdots \\ \mathbf{h}^T \underline{\underline{\mathbf{v}}}_{m,k-q-\left\lfloor \frac{L-1-q}{D} \right\rfloor D} - \hat{\mathbf{h}}_{m,\frac{k-q-\left\lfloor \frac{L-1-q}{D} \right\rfloor D}{D}}^H \underline{\mathbf{v}}_{m,k-q-\left\lfloor \frac{L-1-q}{D} \right\rfloor D}^{D} \end{bmatrix} \cdot \\ & \left[\underline{\underline{\mathbf{v}}}_{n,k-q}^H \mathbf{h} - \underline{\mathbf{v}}_{n,k-q}^{D\,H} \hat{\mathbf{h}}_{n,\frac{k-q}{D}} \,\middle|\, \underline{\underline{\mathbf{v}}}_{n,k-q-D}^H \mathbf{h} - \underline{\mathbf{v}}_{n,k-q-D}^{D\,H} \hat{\mathbf{h}}_{n,\frac{k-q-D}{D}} \,\middle|\, \cdots \,\middle|\, \underline{\underline{\mathbf{v}}}_{n,k-q-\left\lfloor \frac{L-1-q}{D} \right\rfloor D}^H \mathbf{h} - \underline{\mathbf{v}}_{n,k-q-\left\lfloor \frac{L-1-q}{D} \right\rfloor D}^{D\,H} \hat{\mathbf{h}}_{n,\frac{k-q-\left\lfloor \frac{L-1-q}{D} \right\rfloor D}{D}}\right]\Bigg] \mathbf{g}_{n,q}^{*\,D}. \end{aligned} \tag{22}$$

Expanding (22) we arrive at:

$$
\mathrm{E}\left[b_{m,k}b_{n,k}^{*}\right] = \mathbf{g}_{m,{}_{q}^{D}}^{T}\begin{bmatrix} \rho_{0,0} & \rho_{0,1} & \cdots & \rho_{0,\left\lfloor\frac{L-1-q}{D}\right\rfloor} \\ \rho_{1,0} & \rho_{1,1} & \cdots & \rho_{1,\left\lfloor\frac{L-1-q}{D}\right\rfloor} \\ \vdots & \vdots & \vdots & \vdots \\ \rho_{\left\lfloor\frac{L-1-q}{D}\right\rfloor,0} & \rho_{\left\lfloor\frac{L-1-q}{D}\right\rfloor,1} & \cdots & \rho_{\left\lfloor\frac{L-1-q}{D}\right\rfloor,\left\lfloor\frac{L-1-q}{D}\right\rfloor} \end{bmatrix}\mathbf{g}_{n,{}_{q}^{D}}^{*}
$$

$$
= \mathbf{g}_{m,{}_{q}^{D}}^{T}\,\mathbf{P}\,\mathbf{g}_{n,{}_{q}^{D}}^{*} \tag{23}
$$

where

$$
\rho_{s,t} = \mathrm{E}\left[\left(\mathbf{h}^{T}\underline{\underline{\mathbf{v}}}_{m,k-q-sD} - \hat{\mathbf{h}}_{m,\frac{k-q-sD}{D}}^{H}\underline{\mathbf{v}}_{m,{}_{k-q-sD}^{D}}\right)\left(\underline{\underline{\mathbf{v}}}_{n,k-q-tD}^{H}\mathbf{h} - \underline{\mathbf{v}}_{n,{}_{k-q-tD}^{D}}^{H}\hat{\mathbf{h}}_{n,\frac{k-q-tD}{D}}\right)\right]
$$

$$
= \mathrm{E}\Bigg[\underbrace{\mathbf{h}^{T}\underline{\underline{\mathbf{v}}}_{m,k-q-sD}\underline{\underline{\mathbf{v}}}_{n,k-q-tD}^{H}\mathbf{h}}_{\text{Term 1}} - \underbrace{\mathbf{h}^{T}\underline{\underline{\mathbf{v}}}_{m,k-q-sD}\underline{\mathbf{v}}_{n,{}_{k-q-tD}^{D}}^{H}\hat{\mathbf{h}}_{n,\frac{k-q-tD}{D}}}_{\text{Term 2}} -
$$

$$
\underbrace{\hat{\mathbf{h}}_{m,\frac{k-q-sD}{D}}^{H}\underline{\mathbf{v}}_{m,{}_{k-q-sD}^{D}}\underline{\underline{\mathbf{v}}}_{n,k-q-tD}^{H}\mathbf{h}}_{\text{Term 3}} + \underbrace{\hat{\mathbf{h}}_{m,\frac{k-q-sD}{D}}^{H}\underline{\mathbf{v}}_{m,{}_{k-q-sD}^{D}}\underline{\mathbf{v}}_{n,{}_{k-q-tD}^{D}}^{H}\hat{\mathbf{h}}_{n,\frac{k-q-tD}{D}}}_{\text{Term 4}}\Bigg] . \tag{24}
$$

We finally substitute

$$
\begin{aligned} v_{m,k-q} &= \mathbf{f}_{m}^{T}\mathbf{x}_{k-q} \\ &= \mathbf{x}_{k-q}^{T}\mathbf{f}_{m} \end{aligned} \tag{25}
$$

where

$$
\mathbf{f}_{m} = \left[f_{m,0} \;\cdots\; f_{m,L-1}\right]^{T} \tag{26}
$$

and

$$
\mathbf{x}_{k} = \left[x_{k} \;\cdots\; x_{k-L+1}\right]^{T} \tag{27}
$$

into (17) and (21) and the result into (24), apply the linearity property of the expectation, and examine each of the resulting four terms:

$$
\mathrm{E}\left[\mathbf{h}^{T}\underline{\underline{\mathbf{v}}}_{m,k-q-sD}\underline{\underline{\mathbf{v}}}_{n,k-q-tD}^{H}\mathbf{h}\right] = \mathbf{h}^{T}\,\mathrm{E}\left[\begin{bmatrix}\mathbf{f}_{m}^{T}\mathbf{x}_{k-q-sD} \\ \vdots \\ \mathbf{f}_{m}^{T}\mathbf{x}_{k-q-sD-H+1}\end{bmatrix}\left[\mathbf{x}_{k-q-tD}^{T}\mathbf{f}_{n}^{*} \cdots \mathbf{x}_{k-q-tD-H+1}^{T}\mathbf{f}_{n}^{*}\right]\right]\mathbf{h}
$$

$$
= \mathbf{h}^{T}\begin{bmatrix} r_{(t-s)D} & r_{(t-s)D+1} & \cdots & r_{(t-s)D+H-1} \\ r_{(t-s)D-1} & r_{(t-s)D} & \ddots & \vdots \\ \vdots & \ddots & \ddots & r_{(t-s)D+1} \\ r_{(t-s)D-H+1} & \cdots & r_{(t-s)D-1} & r_{(t-s)D} \end{bmatrix}\mathbf{h}
$$

$$
= \mathbf{h}^{T}\mathbf{P}_{1,t-s}\mathbf{h} \quad , \tag{28}
$$

$$
\mathrm{E}\left[\mathbf{h}^{T}\underline{\underline{\mathbf{v}}}_{m,k-q-sD}\underline{\mathbf{v}}_{n,{}_{k-q-tD}^{D}}^{H}\hat{\mathbf{h}}_{n,\frac{k-q-tD}{D}}\right] = \mathbf{h}^{T}\,\mathrm{E}\left[\begin{bmatrix}\mathbf{f}_{m}^{T}\mathbf{x}_{k-q-sD} \\ \vdots \\ \mathbf{f}_{m}^{T}\mathbf{x}_{k-q-sD-H+1}\end{bmatrix}\left[\mathbf{x}_{k-q-tD}^{T}\mathbf{f}_{n}^{*} \cdots \mathbf{x}_{k-q-tD-(N-1)D}^{T}\mathbf{f}_{n}^{*}\right]\right]\hat{\mathbf{h}}_{n,\frac{k-q-tD}{D}}
$$

$$
= \mathbf{h}^{T}\begin{bmatrix} r_{(t-s)D} & r_{(t-s)D+D} & \cdots & r_{(t-s)D+(N-1)D} \\ r_{(t-s)D-1} & r_{(t-s)D+D-1} & \cdots & r_{(t-s)D+(N-1)D-1} \\ \vdots & \vdots & \vdots & \vdots \\ r_{(t-s)D-H+1} & r_{(t-s)D+D-H+1} & \cdots & r_{(t-s)D+(N-1)D-H+1} \end{bmatrix}\hat{\mathbf{h}}_{n,\frac{k-q-tD}{D}}
$$

$$
= \mathbf{h}^{T}\mathbf{P}_{2,t-s}\hat{\mathbf{h}}_{n,\frac{k-q-tD}{D}} \quad , \tag{29}
$$

$$
\mathrm{E}\left[\hat{\mathbf{h}}_{m,\frac{k-q-sD}{D}}^{H}\underline{\mathbf{v}}_{m,{}_{k-q-sD}^{D}}\underline{\mathbf{v}}_{n,{}_{k-q-tD}^{D}}^{H}\mathbf{h}\right] = \hat{\mathbf{h}}_{m,\frac{k-q-sD}{D}}^{H}\,\mathrm{E}\left[\begin{bmatrix}\mathbf{f}_{m}^{T}\mathbf{x}_{k-q-sD} \\ \vdots \\ \mathbf{f}_{m}^{T}\mathbf{x}_{k-q-sD-(N-1)D}\end{bmatrix}\left[\mathbf{x}_{k-q-tD}^{T}\mathbf{f}_{n}^{*} \cdots \mathbf{x}_{k-q-tD-H+1}^{T}\mathbf{f}_{n}^{*}\right]\right]\mathbf{h}
$$

$$
= \hat{\mathbf{h}}_{m,\frac{k-q-sD}{D}}^{H}\begin{bmatrix} r_{(t-s)D} & r_{(t-s)D+1} & \cdots & r_{(t-s)D+H-1} \\ r_{(t-s)D-D} & r_{(t-s)D-D+1} & \cdots & r_{(t-s)D-D+H-1} \\ \vdots & \vdots & \vdots & \vdots \\ r_{(t-s)D-(N-1)D} & r_{(t-s)D-(N-1)D+1} & \cdots & r_{(t-s)D-(N-1)D+H-1} \end{bmatrix}\mathbf{h}
$$

$$
= \hat{\mathbf{h}}_{m,\frac{k-q-sD}{D}}^{H}\mathbf{P}_{3,t-s}\mathbf{h} \quad , \tag{30}
$$

and

$$
\mathrm{E}\left[\hat{\mathbf{h}}_{m,\frac{k-q-sD}{D}}^{H}\underline{\mathbf{v}}_{m,{}_{k-q-sD}^{D}}\underline{\mathbf{v}}_{n,{}_{k-q-tD}^{D}}^{H}\hat{\mathbf{h}}_{n,\frac{k-q-tD}{D}}\right] = \hat{\mathbf{h}}_{m,\frac{k-q-sD}{D}}^{H}\,\mathrm{E}\left[\begin{bmatrix}\mathbf{f}_{m}^{T}\mathbf{x}_{k-q-sD} \\ \vdots \\ \mathbf{f}_{m}^{T}\mathbf{x}_{k-q-sD-(N-1)D}\end{bmatrix}\left[\mathbf{x}_{k-q-tD}^{T}\mathbf{f}_{n}^{*} \cdots \mathbf{x}_{k-q-tD-(N-1)D}^{T}\mathbf{f}_{n}^{*}\right]\right]\hat{\mathbf{h}}_{n,\frac{k-q-tD}{D}}
$$

$$
= \hat{\mathbf{h}}_{m,\frac{k-q-sD}{D}}^{H}\begin{bmatrix} r_{(t-s)D} & r_{(t-s)D+D} & \cdots & r_{(t-s)D+(N-1)D} \\ r_{(t-s)D-D} & r_{(s-t)D} & \ddots & \vdots \\ \vdots & \ddots & \ddots & r_{(t-s)D+D} \\ r_{(t-s)D-(N-1)D} & \cdots & r_{(t-s)D-D} & r_{(t-s)D} \end{bmatrix}\hat{\mathbf{h}}_{n,\frac{k-q-tD}{D}}
$$

$$
= \hat{\mathbf{h}}_{m,\frac{k-q-sD}{D}}^{H}\mathbf{P}_{4,t-s}\hat{\mathbf{h}}_{n,\frac{k-q-tD}{D}} \quad , \tag{31}
$$

where

$$
r_{l} = \begin{cases} \left[\underbrace{0 \;\cdots\; 0}_{-l}\; f_{m,0} \;\cdots\; f_{m,L-1+l}\right]\mathbf{f}_{n}^{*}, & -L+1 \le l \le -1 \\ \left[f_{m,l} \;\cdots\; f_{m,L-1}\; \underbrace{0 \;\cdots\; 0}_{l}\right]\mathbf{f}_{n}^{*}, & 0 \le l \le L-1 \\ 0 & \text{, otherwise} \end{cases} . \tag{32}
$$

Putting this all together we have for (24):

$$
\begin{aligned} \rho_{s,t} = {} & \mathbf{h}^{T}\mathbf{P}_{1,t-s}\mathbf{h} - \mathbf{h}^{T}\mathbf{P}_{2,t-s}\hat{\mathbf{h}}_{n,\frac{k-q-tD}{D}} - \\ & \hat{\mathbf{h}}_{m,\frac{k-q-sD}{D}}^{H}\mathbf{P}_{3,t-s}\mathbf{h} + \hat{\mathbf{h}}_{m,\frac{k-q-sD}{D}}^{H}\mathbf{P}_{4,t-s}\hat{\mathbf{h}}_{n,\frac{k-q-tD}{D}} \end{aligned} \tag{33}
$$

where $\mathbf{P}_{1,t-s}$, $\mathbf{P}_{2,t-s}$, $\mathbf{P}_{3,t-s}$, and $\mathbf{P}_{4,t-s}$ are taken from (28), (29), (30), and (31) respectively.

There are thus two main steps in the procedure for computing ξ_{k}. The first step is to do the calculations in (33) so as to build $\mathbf{P}$ in (23) and calculate $\mathrm{E}\left[b_{m,k}b_{n,k}^{*}\right]$. We must then repeat this first step for all m and n in (3). This is of course for only a single sample index of the MSE!

3. ADAPTIVE FILTER CALCULATIONS

In this section we compute the expected value of the LMS adaptive filter for subband m at sample index k under zero mean, unit variance white Gaussian noise. We begin with the time evolution formula for an adaptive filter [4]:

$$
\hat{\mathbf{h}}_{m,k} = \left(\mathbf{I} - \mu\mathbf{R}_{m}\right)^{k}\left(\hat{\mathbf{h}}_{m,0} - \mathbf{h}_{m,\mathrm{opt}}\right) + \mathbf{h}_{m,\mathrm{opt}} \tag{34}
$$

where $\mathbf{R}_{m}$ is the input correlation matrix for the m^{th} subband adaptive filter, $\hat{\mathbf{h}}_{m,0}$ is the initial setting of the adaptive filter, and $\mathbf{h}_{m,\text{opt}}$ is the Wiener filter solution. We compute $\mathbf{R}_m$ defined as (see Fig. 1):

$$\mathbf{R}_m = \mathrm{E}\left[\mathbf{x}_{m,k}\mathbf{x}_{m,k}^H\right] = \begin{bmatrix} r_{m,0} & r_{m,1} & \cdots & r_{m,N-1} \\ r_{m,1}^* & r_{m,0} & \ddots & \vdots \\ \vdots & \ddots & \ddots & r_{m,1} \\ r_{m,N-1}^* & \cdots & r_{m,1}^* & r_{m,0} \end{bmatrix} \tag{35}$$

where

$$\begin{aligned} r_{m,l} &= \mathrm{E}\left[x_{m,k}x_{m,k-l}^*\right] \\ &= \mathrm{E}\left[v_{m,Dk}v_{m,D(k-l)}^*\right] \\ &= \mathrm{E}\left[\mathbf{f}_m^T\mathbf{x}_{Dk}\mathbf{x}_{D(k-l)}^T\mathbf{f}_m^*\right] \\ &= \begin{cases} \left[\left[f_{m,Dl} \cdots f_{m,L-1} \underbrace{0 \cdots 0}_{Dl}\right]\mathbf{f}_n, & 0 \le Dl \le L-1 \right. \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{36}$$

We next compute the Wiener filter solution for the subband adaptive filter given by:

$$\mathbf{h}_{m,\text{opt}} = \mathbf{R}_m^{-1}\mathbf{p}_m . \tag{37}$$

where $\mathbf{p}_m$ is the cross-correlation between the subband input and the subband reference (desired) signal. Thus we compute

$$\begin{aligned} \mathbf{p}_m &= \mathrm{E}\left[\mathbf{x}_{m,k}y_{m,k}^*\right] \\ &= \left[p_{m,0} \;\; p_{m,-1} \;\; \cdots \;\; p_{m,1-N}\right]^T \end{aligned} \tag{38}$$

where

$$\begin{aligned} p_{-l} &= \mathrm{E}\left[x_{m,k-l}y_{m,k}^*\right] \\ &= \mathrm{E}\left[v_{m,D(k-l)}w_{m,Dk}^*\right] \\ &= \mathrm{E}\left[\mathbf{f}_m^T\mathbf{x}_{D(k-l)}\mathbf{v}_{m,Dk}^H\mathbf{h}\right] \\ &= \mathbf{f}_m^T\,\mathrm{E}\left[\mathbf{x}_{D(k-l)}\left[v_{m,Dk}^* \;\; v_{m,Dk-1}^* \cdots v_{m,Dk-H+1}^*\right]\right]\mathbf{h} \\ &= \mathbf{f}_m^T\,\mathrm{E}\left[\mathbf{x}_{D(k-l)}\left[\mathbf{x}_{Dk}^T\mathbf{f}_m^* \,|\, \mathbf{x}_{Dk-1}^T\mathbf{f}_m^* \,|\cdots|\, \mathbf{x}_{Dk-H+1}^T\mathbf{f}_m^*\right]\right]\mathbf{h} . \end{aligned} \tag{39}$$

Expanding (39) we arrive at:

$$\begin{aligned} p_{-l} &= \mathbf{f}_m^T\,\mathrm{E}\left[\mathbf{x}_{D(k-l)}\left[\mathbf{x}_{Dk}^T\mathbf{f}_m^* \,|\, \mathbf{x}_{Dk-1}^T\mathbf{f}_m^* \,|\cdots|\, \mathbf{x}_{Dk-H+1}^T\mathbf{f}_m^*\right]\right]\mathbf{h} \\ &= \mathbf{f}_m^T\,\mathrm{E}\begin{bmatrix} x_{D(k-l)}\mathbf{x}_{Dk}^T\mathbf{f}_m^* & x_{D(k-l)}\mathbf{x}_{Dk-1}^T\mathbf{f}_m^* & \cdots & x_{D(k-l)}\mathbf{x}_{Dk-H+1}^T\mathbf{f}_m^* \\ x_{D(k-l)-1}\mathbf{x}_{Dk}^T\mathbf{f}_m^* & x_{D(k-l)-1}\mathbf{x}_{Dk-1}^T\mathbf{f}_m^* & \cdots & x_{D(k-l)-1}\mathbf{x}_{Dk-H+1}^T\mathbf{f}_m^* \\ \vdots & \vdots & \ddots & \vdots \\ x_{D(k-l)-L+1}\mathbf{x}_{Dk}^T\mathbf{f}_m^* & x_{D(k-l)-L+1}\mathbf{x}_{Dk-1}^T\mathbf{f}_m^* & \cdots & x_{D(k-l)-L+1}\mathbf{x}_{Dk-H+1}^T\mathbf{f}_m^* \end{bmatrix}\mathbf{h} \end{aligned}$$

Column: 1, 2, Dl, $Dl+1$, $Dl+2$, $Dl+L$, $Dl+L+1$, H

$$= \mathbf{f}_m^T \left[\begin{array}{cccccccc|ccc} f_{m,Dl}^* & f_{m,Dl-1}^* & \cdots & f_{m,1}^* & f_{m,0}^* & 0 & \cdots & 0 & 0 & \cdots & 0 \\ f_{m,Dl+1}^* & f_{m,Dl}^* & \cdots & f_{m,2}^* & f_{m,1}^* & f_{m,0}^* & & & & & \\ \vdots & f_{m,Dl+1}^* & & f_{m,3}^* & f_{m,2}^* & f_{m,1}^* & & & & & \\ f_{m,L-1}^* & \vdots & & & & & & \vdots & & & \\ 0 & f_{m,L-1}^* & & \vdots & \vdots & \vdots & & & \vdots & & \vdots \\ \vdots & 0 & \ddots & & & & & & & & \\ \vdots & \vdots & \ddots & f_{m,L-1}^* & f_{m,L-2}^* & & & 0 & & & \\ 0 & 0 & \cdots & 0 & f_{m,L-1}^* & f_{m,L-2}^* & \cdots & f_{m,0}^* & 0 & \cdots & 0 \end{array}\right]\mathbf{h} . \tag{40}$$

4. CONCLUSION

In this paper we have calculated the reconstructed MSE for a general, LMS subband adaptive filtering system under zero mean, unit variance white Gaussian noise. In turn this calculation enables us to predict performance as measured by the MSE for a wide variety of subband adaptive filtering systems. These systems include both critically sampled and oversampled systems, systems with increased bandwidth analysis filters, systems with perfect reconstruction filter banks, systems with variable length adaptive filters, etc....

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